ON SUPERMANIFOLDS ASSOCIATED WITH THE COTANGENT BUNDLE (Lie Groups, Geometric Structures and Differential Equations : One Hundred Years after Sophus Lie)

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INTRODUCTION

We study here the problem of classification of complex analytic supermanifolds. Clearly, with any holomorphic vector bundle $\mathbf{E}$ over a complex manifold M one can associate the so-called split supermanifold $(M, \bigwedge \mathcal{E})$, where $\mathcal{E}$ is the sheaf of holomorphic sections of $\mathbf{E}$. On the other hand, each supermanifold $(M, \mathcal{O})$ can be deformed into a split one which is called the retract of $(M, \mathcal{O})$. Thus, our problem is reduced to the problem of classification of holomorphic vector bundles and to the problem of classification of complex analytic supermanifolds with a given retract. We give here a survey of results concerning the second problem. We consider the case when $\mathbf{E} = \mathbf{T}(M)^*$ is the cotangent bundle of M, though some important facts exposed in Sections 1 and 3 are valid in the general case. Thus, we deal mainly with the problem of classification of complex supermanifolds with retract (M, Ω), where Ω is the sheaf of holomorphic forms on a complex manifold M.

Section 1 contains necessary definitions and some preliminary facts, including the theorem of Green reducing our classification problem to a problem of non-abelian cohomology theory. In Section 2 we give a direct construction of supermanifolds with retract (M, Ω) starting from a d-closed $(1,1)$-form or from a holomorphic line bundle on M (see [11]). In particular, we see that for any compact Kähler manifold M with $\dim M > 1$ there exist non-split supermanifolds of this sort. In Section 3 we construct a non-abelian cochain complex in the sense of [8, 12], whose 1-cohomology set gives a solution of our problem. This complex is actually of a type considered by Nijenhuis and Richardson [7] in connection with the deformation theory of algebras, i.e., it is related to a differential graded Lie superalgebra. The corresponding differential Lie superalgebra was introduced in [10]; its elements are derivations of the sheaf of smooth differential forms on M. For a compact manifold M, our complex gives rise to a finite-dimensional affine algebraic variety which can serve as a moduli variety for our classification problem; it is analogous to the Kuranishi family of complex structures on a compact manifold (see [5]). The detailed exposition of this theory see in [13, 15]. Section 4 contains applications to the case when M is a flag manifold.

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1. Complex supermanifolds

We consider here complex analytic supermanifolds, i.e., $\mathbb{Z}_2$-graded ringed spaces $(M, \mathcal{O})$ locally isomorphic to $(\tilde{U}, \bigwedge_{\mathcal{F}_n}(\xi_1, \dots, \xi_m))$, where $\tilde{U}$ is an open subset of $\mathbb{C}^n$ and $\mathcal{F}_n$ the sheaf of holomorphic functions in $\mathbb{C}^n$ and the exterior algebra sheaf $\mathcal{F}_{n|m} = \bigwedge_{\mathcal{F}_n}(\xi_1, \dots, \xi_m)$ is $\mathbb{Z}_2$-graded in the usual way. Such a local isomorphism gives us a *chart* on an open subset $U \subset M$. The coordinates $z_1, \dots, z_n$ of $\mathbb{C}^n$ are called *even local coordinates* on U, while $\xi_1, \dots, \xi_m$ are called *odd* ones. If U and V are two open subsets of M admitting two charts with local coordinates x_i $(i = 1, \dots, n)$, ξ_j $(j = 1, \dots, m)$ and y_i $(i = 1, \dots, n)$, η_j $(j = 1, \dots, m)$, then in $U \cap V$ we can write

$$
\begin{aligned}
y_i &= \varphi_i(x_1, \dots, x_n, \xi_1, \dots, \xi_m), \quad i = 1, \dots, n; \\
\eta_j &= \psi_j(x_1, \dots, x_n, \xi_1, \dots, \xi_m), \quad j = 1, \dots, m,
\end{aligned}
\tag{1}
$$

where φ_i, ψ_j are, respectively, even and odd sections of $\mathcal{F}_{n|m}$ called the *transition functions*. We write $\dim(M, \mathcal{O}) = n|m$.

Here is a classical example of a complex supermanifold. Let M be a complex manifold of dimension n. By definition, this is a ringed space $(M, \mathcal{F})$, where $\mathcal{F}$ is the sheaf of holomorphic functions on M. Extending this sheaf to the sheaf $\Omega = \bigoplus_{p=0}^{n} \Omega^p$ of holomorphic exterior forms on M, we get the graded ringed space (M, Ω). This is a supermanifold of dimension $n|n$. In fact, let U be an open subset of M, where a chart with local coordinates $x_1, \dots, x_n$ is defined. Clearly, the sheaf $\Omega|U$ can be identified with $\bigwedge_{\mathcal{F}_n}(dx_1, \dots, dx_n)$. Denoting $\xi_j = dx_j$, we see that x_i, ξ_j are local coordinates for (M, Ω). If V is another open subset with local coordinates y_i and $\eta_j = dy_j$, then the transition functions in $U \cap V$ have the form

$$
\begin{aligned}
y_i &= \varphi_i(x_1, \dots, x_n), \quad i = 1, \dots, n, \\
\eta_j &= \sum_{k=1}^{n} \frac{\partial y_j}{\partial x_k} \xi_k, \quad j = 1, \dots, n,
\end{aligned}
\tag{2}
$$

where φ_i are the usual transition functions for M.

The transition functions (2) are very simple: y_i do not depend on ξ_j, while η_j contain only terms of degree 1 in ξ_j. We express this fact by saying that (M, Ω) is a *split* complex supermanifold. Quite similarly, we may associate a split complex supermanifold with any holomorphic vector bundle $\mathbf{E}$ over a complex manifold M; our example corresponds to the case $\mathbf{E} = \mathbf{T}(M)^*$ (the cotangent bundle).

Consider now the following problem: how can we add to (2) additional terms of degrees 2, 4 etc. for y_i and of degrees 1, 3 etc. for η_j, in order to get a supermanifold structure on M, whose structure sheaf $\mathcal{O}$ is not isomorphic to Ω? The supermanifolds obtained in this way are called *non-split supermanifolds with retract* (M, Ω), and we would like to classify them up to isomorphism. A similar problem can be posed for an arbitrary holomorphic vector bundle $\mathbf{E}$.

For the complex grassmannians $M = \mathrm{Gr}_{n,k}$ (and more generally, for complex manifolds of flags), examples of supermanifolds with retract (M, Ω) were given by Manin. These are the so-called *Π-symmetric supergrassmannians* $\Pi\,\mathrm{Gr}_{n|n,k|k}$ defined in [6]. It is proved in [9] that $\Pi\,\mathrm{Gr}_{n|n,k|k}$ is non-split whenever $n > 2$.

The supermanifolds with a given retract can be classified in terms of the 1-cohomology with values in an automorphism sheaf of the structure sheaf of the retract. In our case, consider the sheaf $\mathcal{A}ut_{(2)}\Omega$ of automorphisms a of the $\mathbb{Z}_2$-graded algebra sheaf Ω such that $a(\psi) - \psi \in \bigoplus_{p\geq 2} \Omega^p$ for any $\psi \in \Omega$. The group $\operatorname{Aut} \mathbf{T}(M)^*$ acts on the automorphism sheaf of Ω by inner automorphisms leaving invariant the subsheaf $\mathcal{A}ut_{(2)}\Omega$ and hence on the 1-cohomology of this subsheaf. If $(M, \mathcal{O})$ is a supermanifold with retract (M, Ω), then we may assume that its transition functions (1) have the functions (2) as their first terms and thus are obtained from (2) by an automorphism $g_{UV} \in \Gamma(U \cap V, \mathcal{A}ut_{(2)}\Omega)$. The following theorem (in a more general form) was proved by Green [2].

Theorem 1.1. *The automorphisms g_{UV} form a Čech 1-cocycle of an open cover of M with values in the sheaf $\mathcal{A}ut_{(2)}\Omega$. This correspondence gives rise to a bijection between the isomorphy classes of supermanifolds with retract (M, Ω) and the orbits of the group* $\operatorname{Aut} \mathbf{T}(M)^*$ *on $H^1(M, \mathcal{A}ut_{(2)}\Omega)$ under the action described above. The split supermanifold (M, Ω) corresponds to the unit element $e \in H^1(M, \mathcal{A}ut_{(2)}\Omega)$.*

For an arbitrary complex supermanifold $(M, \mathcal{O})$, denote by $\mathcal{T} = \mathcal{D}er\,\mathcal{O}$ the sheaf of derivations of the structure sheaf $\mathcal{O}$. The sheaf $\mathcal{T}$ is called the *tangent sheaf* of M. The tangent sheaf is in a natural way a sheaf of $\mathbb{Z}_2$-graded left $\mathcal{O}$-modules. On the other hand, it can be regarded as a sheaf of complex Lie superalgebras under the bracket

$$[u, v] = uv + (-1)^{p(u)p(v)+1} vu. \tag{3}$$

Sections of $\mathcal{T}$ (*holomorphic vector fields* on $(M, \mathcal{O})$) form the Lie superalgebra $\mathfrak{v}(M, \mathcal{O}) = \Gamma(M, \mathcal{T})$; it is finite-dimensional whenever M is compact.

In what follows, we shall use the cohomology groups $H^p(M, \mathcal{T})$ with values in the tangent sheaf; they are finite-dimensional vector spaces whenever M is compact. The bracket (3) induces a bracket in $H^*(M, \mathcal{T}) = \bigoplus_{p\geq 0} H^p(M, \mathcal{T})$ giving a graded Lie superalgebra that contains $H^0(M, \mathcal{T}) = \mathfrak{v}(M, \mathcal{O})$ as a subalgebra.

If the supermanifold $(M, \mathcal{O})$ is split, then $\mathcal{T} = \bigoplus_{p\geq -1} \mathcal{T}_p$ is a $\mathbb{Z}$-graded sheaf of Lie superalgebras. E.g., for $\mathcal{O} = \Omega$ the grading is given by

$$\mathcal{T}_p = \mathcal{D}er_p\Omega = \{v \in \mathcal{T} \mid v(\Omega^q) \subset \Omega^{q+p} \text{ for all } q \in \mathbb{Z}\}.$$

The structure of the sheaf $\mathcal{T} = \mathcal{D}er\,\Omega$ is described by the following theorem proved essentially by Frölicher and Nijenhuis [1].

Theorem 1.2. *There is the following exact sequence of locally free analytic sheaves on M:*

$$0 \to \Omega^{p+1} \otimes \Theta \xrightarrow{i} \mathcal{T}_p \xrightarrow{\alpha} \Omega^p \otimes \Theta \to 0,$$

Here $\Theta = \mathcal{D}er\mathcal{F}$ is the tangent sheaf of the manifold M, the mapping α is the restriction of a derivation of degree p onto the subsheaf $\mathcal{F}$, and i identifies any sheaf homomorphism $\Omega^1 \to \Omega^{p+1}$ with a derivation of degree p that is zero on $\mathcal{F}$.

This sequence is split, the splitting mapping $l : \Omega \otimes \Theta \to \mathcal{T}$ being defined by

$$l(\varphi) = [i(\varphi), d],$$

where d is the exterior derivative regarded as a section of $\mathcal{T}_1$.

Corollary. *There is the following decomposition into the direct sum of sheaves of vector spaces:*

$$\mathcal{T} = i(\Omega \otimes \Theta) \oplus l(\Omega \otimes \Theta).$$

Note that $\Omega \otimes \Theta$ is the so-called sheaf of holomorphic *vector-valued forms*. Also, for $p = 0$ the derivation $l(u)$, $u \in \Theta$, is the classical Lie derivative along the vector field u.

As in the classical Lie theory, there exists a natural relationship between automorphisms and derivations of the sheaf Ω (see [16]). Let us denote

$$\mathcal{T}_{\bar{0}(2p)} = \bigoplus_{k \geq p} \mathcal{T}_{2k}.$$

Then we have the exponential mapping

$$\exp : \mathcal{T}_{\bar{0}(2)} \to \mathcal{A}ut_{(2)}\Omega.$$

It is expressed by the usual exponential series which is actually a polynomial, since any $v \in \mathcal{T}_{\bar{0}(2)}$ satisfies $v^k = 0$ for any $k > \left[\frac{m}{2}\right]$. One proves that exp is bijective. Thus it is an isomorphism of sheaves of sets (but in general not of groups). We denote $\log = \exp^{-1}$. One proves that

$$\lambda_2 : \mathcal{A}ut_{(2)}\Omega \to \mathcal{T}_2, \tag{4}$$

where $\lambda_2(a)$ is the 2-component of $\log a \in \mathcal{T}_{\bar{0}(2)}$, is a homomorphism of sheaves of groups.

2. A construction of non-split supermanifolds

Here we give a direct construction of non-split supermanifolds with retract (M, Ω) (see [11]). Let $\mathcal{Z}\Omega^1$ denote the subsheaf of Ω^1 consisting of closed forms and $\beta : \mathcal{Z}\Omega^1 \to \Omega^1$ the inclusion mapping. Consider the mapping $\mu : \mathcal{Z}\Omega^1 \to \mathcal{A}ut_{(2)}\Omega$ given by

$$\mu(\psi) = \exp(\psi d) = \mathrm{id} + \psi d, \quad \psi \in \mathcal{Z}\Omega^1.$$

One verifies easily that this is a homomorphism of sheaves of groups. It follows that we have the cohomology homomorphism (i.e. a mapping, taking 0 to the unit element)

$$\mu^* : H^1(M, \mathcal{Z}\Omega^1) \to H^1(M, \mathcal{A}ut_{(2)}\Omega).$$

Using Theorem 1.2 and the homomorphism λ_2 given by (4), we get

Proposition 2.1. *Suppose that* $\dim M > 1$ *and that* $\zeta, \zeta' \in H^1(M, \mathcal{Z}\Omega^1)$. *If* $\mu^*(\zeta) = \mu^*(\zeta')$, *then* $\beta^*(\zeta) = \beta^*(\zeta')$.

Let $\mathfrak{U} = (U, V, \dots)$ be an open cover of M and let $\psi = (\psi_{UV})$ be a cocycle from $Z^1(\mathfrak{U}, \mathcal{Z}\Omega^1)$. Then the above construction assignes to ψ the supermanifold given by the cocycle $g = (g_{UV}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\Omega)$, where

$$g_{UV} = \mathrm{id} + \psi_{UV} d. \tag{5}$$

Due to Theorem 1.1, we see from Proposition 2.1 that this supermanifold is non-split if and only if the cohomology class of ψ in $H^1(M, \Omega^1)$ is non-zero.

Now we pass to an important case, where a "closed cocycle" ψ appears. Let ω be a (1,1)-form on M satisfying $d\omega = 0$. Then, clearly, $\bar\partial\omega = 0$, and hence ω determines a Dolbeault cohomology class $[\omega] \in H^{1,1}(M, \mathbb{C})$. If we denote by $D : H^{1,1}(M,\mathbb{C}) \to H^1(M,\Omega^1)$ the Dolbeault isomorphism, then it turns out that $D([\omega])$ can be represented by a closed Čech cocycle. Denote by $\Phi^{p,q}$ the sheaf of smooth complex-valued (p,q)-forms on M. Then we have the exact sequence of sheaves:

$$0 \to \mathcal{Z}\Omega^1 \to \Phi^{1,0}_{\partial} \xrightarrow{\bar\partial} \mathcal{Z}\Phi^{1,1} \to 0,$$

where $\Phi^{1,0}_{\partial} \subset \Phi^{1,0}$ is the subsheaf of ∂-closed $(1,0)$-forms and $\mathcal{Z}\Phi^{1,1} \subset \Phi^{1,1}$ the subsheaf of d-closed $(1,1)$-forms. Consider the corresponding connecting homomorphism

$$\delta^* : \Gamma(M, \mathcal{Z}\Phi^{1,1}) \to H^1(M, \mathcal{Z}\Omega^1).$$

Then $\beta^*\delta^*\omega$ is the Dolbeault class of ω. As a result, we get the mapping

$$\mu^* \circ \delta^* : \Gamma(M, \mathcal{Z}\Phi^{1,1}) \to H^1(M, \mathcal{A}ut_{(2)}\Omega).$$

Thus, any $(1,1)$-form ω on M such that $d\omega = 0$ determines a supermanifold with retract (M,Ω). To obtain an expression of the corresponding cocycle g, we consider an open cover $\mathfrak{U} = (U, V, \dots)$ of M such that $\omega = \bar\partial\psi_U$ in any U, where $\psi_U \in \Phi^{1,0}_{\partial}(U)$. Then $\delta^*\omega$ is represented by the cocycle $\psi = (\psi_{UV}) \in Z^1(\mathfrak{U}, \mathcal{Z}\Omega^1)$, where $\psi_{UV} = \psi_V - \psi_U$ in $U \cap V \neq \emptyset$. Finally, the cocycle g is given by (5).

Using Proposition 2.1, we deduce the following result.

Theorem 2.1. *If M is a compact Kähler manifold, then we have a linear mapping $\tilde\delta : H^{1,1}(M,\mathbb{C}) \to H^1(M, \mathcal{Z}\Omega^1)$ such that $\beta^* \circ \tilde\delta = D$. The mapping $\mu^* \circ \tilde\delta : H^{1,1}(M,\mathbb{C}) \to H^1(M, \mathcal{A}ut_{(2)}\Omega)$ is injective, whenever $n > 1$, and takes 0 to e.*

Applying the above construction, we can associate a supermanifold $(M, \mathcal{O})$ with retract (M,Ω) with any holomorphic line bundle L over M. The closed $(1,1)$-form ω will be here the curvature form of a Hermitian metric on L. More precisely, we have the mapping

$$\mu^* \circ \mathfrak{D}^* : \mathrm{Pic}(M) = H^1(M, \mathcal{F}^\times) \to H^1(M, \mathcal{A}ut_{(2)}\Omega)$$

corresponding to the homomorphism of sheaves of groups

$$\mu \circ \mathfrak{D} : \mathcal{F}^\times \to \mathcal{A}ut_{(2)}\Omega,$$

where $\mathfrak{D}$ is the logarithmic differential, i.e.,

$$\mathfrak{D}f = f^{-1}df = d\log f, \quad f \in \mathcal{F}^\times.$$

Let $L \in \mathrm{Pic}(M)$ be given by a cocycle $h = (h_{UV}) \in Z^1(\mathfrak{U}, \mathcal{F}^\times)$. Then $(M, \mathcal{O})$ is determined by the following cocycle $g = (g_{UV}) \in Z^1(M, \mathcal{A}ut_{(2)}\Omega)$:

$$g_{UV} = \mathrm{id} + (h_{UV}^{-1} dh_{UV})d.$$

For example, the canonical line bundle $K_M = \bigwedge^n \mathbf{T}(M)^*$ gives rise to a supermanifold called the *canonical supermanifold* over M. It corresponds to the canonical form defined by Koszul [4] and is not necessarily non-split.

3. A GENERAL CLASSIFICATION THEOREM

We retain the notation of the preceeding sections. Here we are going to express the cohomology set $H^1(M, \mathcal{A}ut_{(2)}\Omega)$ (see Theorem 1.1) in terms of differential forms on M. To do this, we use a non-linear complex similar to the non-linear de Rham and Dolbeault complexes studied, e.g., in [3, 8, 15]. Actually, a general complex of this sort was considered in [7], but it was used there only in the finite-dimensional situation. We consider here the split supermanifold (M, Ω), but the cotangent bundle can be easily replaced by an arbitrary holomorphic vector bundle over M in all general theorems formulated below.

The first step is the construction of a fine resolution of the sheaf $\mathcal{T} = \mathcal{D}er\,\Omega$. Theorem 1.2 implies that $\mathcal{T}$ is a locally free analytic sheaf on M, and hence we can form the standard Dolbeault—Serre resolution of $\mathcal{T}$. More precisely, we set

$$\mathcal{R}_{p,q} = \Phi^{0,q} \otimes \mathcal{T}_p,$$
$$\mathcal{R} = \bigoplus_{p \geq -1, q \geq 0} \mathcal{R}_{p,q},$$
$$\bar{\partial}(\varphi \otimes u) = (\bar{\partial}\varphi) \otimes u, \quad \varphi \in \mathcal{R}_{0,q}, u \in \mathcal{T}_p.$$

Then the sequence

$$0 \to \mathcal{T} \xrightarrow{i} \mathcal{R}_{*,0} \xrightarrow{\bar{\partial}} \mathcal{R}_{*,1} \xrightarrow{\bar{\partial}} \dots \tag{6}$$

is the desired resolution. However, it is convenient to write this resolution in a more complicated form, using derivations of the sheaf Φ of smooth forms. Our purpose is to obtain a resolution endowed with a bracket operation that extends the operation (3) given in $\mathcal{T}$.

Consider the sheaf of graded Lie algebras $\mathcal{D}er\Phi$ and denote

$$\bar{D} = \operatorname{ad}\bar{\partial}.$$

Clearly, $\bar{D}$ is a derivation of bidegree $(0,1)$) of $\mathcal{D}er\Phi$, and

$$\bar{D}^2 = \frac{1}{2}[\bar{D}, \bar{D}] = \frac{1}{2}\operatorname{ad}[\bar{\partial}, \bar{\partial}] = 0.$$

Set

$$\mathcal{S} = \{u \in \mathcal{D}er\Phi \mid u(\bar{f}) = u(d\bar{f}) = 0 \text{ for any } f \in \mathcal{F}\}.$$

One sees readily that $\mathcal{S}$ is a subsheaf of bigraded subalgebras of $\mathcal{D}er\Phi$ that is invariant under $\bar{D}$. Moreover, $\mathcal{T}$ is identified with the kernel of the mapping $\bar{D} : \mathcal{S}_{*,0} \to \mathcal{S}_{*,1}$. Thus, we get the sequence

$$0 \to \mathcal{T} \xrightarrow{i} \mathcal{S}_{*,0} \xrightarrow{\bar{D}} \mathcal{S}_{*,1} \xrightarrow{\bar{D}} \dots. \tag{7}$$

By [10], this is a fine resolution of $\mathcal{T}$ isomorphic to (6). Moreover, i is a homomorphism of graded Lie algebra sheaves, and hence the natural bracket in $\mathcal{S}$ may be used to calculate the bracket in $H^*(M, \mathcal{T})$ induced by the Lie bracket defined in $\mathcal{T}$. We also need the sheaf of groups

$$\mathcal{P}\mathcal{A}ut\Phi = \{a \in \mathcal{A}ut\Phi \mid a(\bar{\psi}) = \bar{\psi} \text{ for all } \psi \in \Omega\}.$$

and its subsheaf

$$\mathcal{P}\mathcal{A}ut_{(2)}\Phi = \{a \in \mathcal{A}ut\Phi \mid a(\psi) - \psi \in \bigoplus_{p \geq 2} \Phi^p,\ \psi \in \Phi\}.$$

The sheaf of groups $\mathcal{P}\mathcal{A}ut_{(2)}\Phi$ acts on $\mathcal{S}$ by the automorphisms $\operatorname{Int} a(u) = aua^{-1}$.

Consider now the triple (K^0, K^1, K^2), where

$$K^0 = \Gamma(M, \mathcal{P}\mathcal{A}ut_{(2)}\Phi), \quad K^p = \bigoplus_{k \geq 2} \Gamma(M, \mathcal{S}_{2k,p}), \quad p = 1, 2,$$

and define the mappings $\delta_0 : K^0 \to K^1$ and $\delta_1 : K^1 \to K^2$ by

$$\begin{aligned} \delta_0(a) &= \bar{\partial} - a\bar{\partial}a^{-1}, \\ \delta_1(u) &= \bar{D}u - \frac{1}{2}[u, u] = -\frac{1}{2}[u - \bar{\partial}, u - \bar{\partial}]. \end{aligned}$$

Clearly, $\delta_1(0) = 0$.

Proposition 3.1.

(1) *The mapping δ_0 is a crossed homomorphism, i.e.,*

$$\delta_0(ab) = \delta_0(a) + a\delta_0(b)a^{-1}, \quad a, b \in K^0.$$

(2) *The corresponding affine action of K^0 on K^1 is given by*

$$\rho(a)(u) \stackrel{\text{def}}{=} \delta_0(a) + aua^{-1} = a(u - \bar{\partial})a^{-1} + \bar{\partial}.$$

(3) *The mapping δ_1 satisfies*

$$\delta_1(\rho(a)(u)) = a\delta_1(u)a^{-1}.$$

This proposition shows that the triple $K = (K^0, K^1, K^2)$ with coboundary mappings δ_p and actions Int of K^0 on K^p, $p = 1, 2$, is a non-abelian cochain complex in the sense of [8, 15]. In particular, we can define its 1-cohomology set

$$H^1(K) = \operatorname{Ker} \delta_1 / \rho$$

with the distinguished point 0. Using the machinery of non-abelian complexes, we get the following result (see [13]).

Theorem 3.1. *We have an isomorphism of sets with distinguished points*

$$\nu : H^1(K) \to H^1(M, \mathcal{A}ut_{(2)}\Omega).$$

The mapping ν can be expressed quite explicitly. Take $z \in K^1$ such that $\delta_1(z) = 0$. There exists an open cover $\mathfrak{U} = (U, V, \ldots)$ of M such that $z = \delta_0(a_U)$, where $a_U \in \Gamma(U, \mathcal{P}\mathcal{A}ut_{(2)}\Phi)$ for any U. Define $b \in Z^1(\mathfrak{U}, \mathcal{P}\mathcal{A}ut_{(2)}\Phi)$ by $b_{UV} = a_U^{-1}a_V$. One sees that b_{UV} preserve the subsheaf $\Omega | U \cap V$, and hence we may regard b as a cocycle from $Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\Omega)$. Then ν maps the cohomology class of z onto that of b.

Example. Without going into details, we show, how to express the construction of Section 2 in terms of the complex K.

Let $\omega \in \Gamma(M, \Phi^{1,1})$ be a (1,1)-form satisfying $d\omega = 0$. Consider the derivation $u = \omega\partial$ of Φ. Clearly, $u \in S_{2,1}$. Moreover, $\bar{D}u = [u, u] = 0$, and hence $\delta_1(u) = 0$. By Theorem 3.1, u determines a cohomology class $\tilde{u} \in H^1(M, \mathcal{A}ut_{(2)}\Omega)$. One sees that $\tilde{u} = \mu^*\delta^*(u)$.

In the case when M is compact, Theorem 3.1 allows to use Hodge theory for constructing a moduli variety for our classification problem (see [13]). This variety is actually an algebraic subvariety of $H^1(M, \bigoplus_{k\geq 1} \mathcal{T}_{2k})$. Note the following simple case when this variety coincides with $H^1(M, \mathcal{T}_2)$.

Proposition 3.2. *If $H^1(M, \mathcal{T}_{2q}) = H^2(M, \mathcal{T}_{2q}) = 0$ for all $q \geq 3$, then λ_2^* : $H^1(M, \mathcal{A}ut_{(2)}\Omega) \to H^1(M, \mathcal{T}_2)$ is an isomorphism.*

This can be deduced from Theorem 3.1 (a more direct proof see in [14]).

4. Applications to flag manifolds

In this section, we consider the case when M is a flag manifold of a connected semisimple complex Lie group G. We may identify M with the coset space G/P, where P is a parabolic subgroup of G. The subgroup P is determined by a subset $S \subset \Pi$, where Π is the system of simple roots of G. E.g., P is maximal whenever $|\Pi \setminus S| = 1$. Let Γ denote the subgroup of $\operatorname{Aut}\Pi$ leaving S invariant. It is known that Γ can be interpreted as a group of biholomorphic transformations of M.

Since M is Kähler, the construction of Section 2 gives rise to a non-empty family of non-split supermanifolds having (M, Ω) as their retract. More precisely, Theorem 2.1 implies

Theorem 4.1. *Let $M = G/P$ is a flag manifold of dimension ≥ 2, where G is simple, and denote $r = |\Pi \setminus S|$. Then there exists a family of distinct non-split supermanifolds parametrized by $\mathbb{CP}^{r-1}/\Gamma$ and having (M, Ω) as their common retract.*

If P is maximal, then this family consists of a unique supermanifold, which is isomorphic to the canonical one.

Now suppose that M is a simply connected irreducible compact Hermitian symmetric space. One proves (see [9]) that the conditions of Proposition 3.2 are satisfied. Moreover, our problem for these manifolds M has the following complete solution.

Theorem 4.2. *Suppose that M is a simply connected irreducible compact Hermitian symmetric space of dimension ≥ 2.*

If $M = \mathrm{Gr}_{n,s}$, $1 < s < n-1$, then non-split supermanifolds with retract (M, Ω) are parametrized by $\mathbb{CP}^1/\Gamma$, where

$$\Gamma = \begin{cases} \mathbb{Z}_2 & \text{if } n = 2s \\ \{e\} & \text{otherwise.} \end{cases}$$

Otherwise, there exists (up to isomorphism) precisely one non-split supermanifold with retract (M, Ω), namely, the canonical one.

It follows that the Π-symmetric supergrassmannian $\Pi\,\mathrm{Gr}_{n|n,k|k}$ is not rigid, except of the case when $k = 1$ or $n-1$, i.e., $M = \mathbb{CP}^{n-1}$.

In [9] the Lie superalgebra $\mathfrak{v}((M,\mathcal{O}))$ for all supermanifolds described in Theorem 4.2 is calculated. It is proved, in particular, that $\Pi\,\mathrm{Gr}_{n|n,k|k}$ is the only homogeneous non-split supermanifold with retract (M,Ω), where M is a simply connected irreducible compact Hermitian symmetric space.

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